FOUNDATION OF SCIENTIFIC COMMUNICATION:
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Exercise: From Poisson to Quantum Evolution

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Foreword

Upon completing this exercise set: you have familiarised yourself with Poisson brackets, matrix commutators and algebras; you will have explicitly worked out the Dirac quantisation; you will have solved the quantum harmonic oscillator explicitly; and you will have derived time evolution within quantum mechanics in several perspectives.

I have tried to make this assignment as accessible as possible, guiding the way when difficult examples arise. In many of the exercises, I have provided the final answer since, truly, the derivation is the most important part of the journey to learn. I hope you, the eager explorer, will find this exercise sheet to be helpful in your quest to understand the world.

∼Remko

1. Poisson algebra

Motivated by the Heisenberg relation Eq.(last eq), we will now explore the mystique of the Poisson brackets. Let $f(x,p,t)$ and $g(x,p,t)$ be two function of time t, time-dependent position coordinates $x(t)$, and momentum coordinates $p(t)$. The Poisson bracket of $f(x,p,t)$ and $g(x,p,t)$ is defined as,

$$\{f,g\} \equiv \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}. \tag{1}$$

To see how this is relevant, we can consider the time derivative of $f(x,p,t)$ using the chain rule,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t}. \tag{2}$$

Using the Hamilton equations Eq. (21), you can write,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial x}. \tag{3}$$

Exercise 1. Write the time derivative $\mathrm{d}f/\mathrm{d}t$ in terms of a Poisson bracket.

If you did it correctly, you should find.[1]

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\}. \tag{4}$$

The Poisson brackets determine the implicit time dependence of functions.

Exercise 2. As a warm-up to working with Poisson brackets, calculate the expressions $\{H, H\}$, $\{x, x\}$, $\{p, p\}$, $\{x, p\}$, $\{x, H\}$ and $\{p, H\}$ explicitly. Use the Hamilton equations Eq.(future Hami) to rewrite the last two Poisson brackets. Additionally, interpret the first expression using Eq.(1+loop) for a time-independent Hamiltonian.

The Poisson brackets satisfy some identities that allow us to upgrade them to a so-called Poisson algebra. For three functions $f(x, p, t)$, $g(x, p, t)$, and $h(x, p, t)$ in this algebra and two real numbers $a, b \in \mathbb{R}$, the identities are given by:

1. **Anticommutativity**,
$$\{f, g\} = -\{g, f\}; \tag{5}$$
2. **Bilinearity**,
$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}; \tag{6}$$
3. **Leibniz's rule**,
$$\{fg, h\} = f\{g, h\} + \{f, h\}g; \tag{7}$$
4. **Jacobi identity**,
$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0. \tag{8}$$

Exercise 3 (*Tedious*)**. Verify explicitly that the Poisson brackets satisfy these algebra relations Eq.** (5)-(8)**.**

From now on, we will forget the definition of the Poisson brackets, and we will only focus on these relations of the Poisson brackets. This is really what it means to be an algebra: we can use the algebraic structure to deduce new information.

Exercise 4. Using only Eq. (5)-(8)**, determine the value of** $\{0, 0\}$[2] **and show a second relation of bilinearity,**

$$\{h, af + bg\} = a\{h, f\} + b\{h, g\} \tag{9}$$

Exercise 5 (*Bonus*)**. Calculate** $\frac{\mathrm{d}}{\mathrm{d}t}\{f, g\}$**. You can do this explicitly or you can be smart about it.**

[1]Notice how we are expressing a part of this equation in terms of an abstract algebra, which means that we could replace this with a different algebra, as long as the identities match. This new algebra could use different objects, so you need to be careful not to end up in some kind of scenario where you're forced to do physics with operators.

[2]Here 0 is understood as a function that is identically zero for all inputs, $0(x, p, t) \equiv 0$.

2. Matrix commutator algebra

With the explicit concept of a Poisson bracket algebra in mind, we will freshly start anew with matrices. Unfortunately, here I need to assume a bit of preknowledge on how matrices work, but I will rapidly recap all that we need. If you are struggling with the topic, I recommend you familiarise yourself with matrices by watching a video or asking help in the discord physics channel.

A matrix[3] $\hat{M}$ is an object of $n \times n$ components m_{ij} arranged neatly into a matrix, more specifically,

$$\hat{M} = \begin{pmatrix} m_{00} & m_{01} & \dots & m_{0n} \\ m_{10} & m_{11} & \dots & m_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n0} & m_{n1} & \dots & m_{nn} \end{pmatrix}. \tag{10}$$

Matrix multiplication is then given by

$$\begin{aligned} \hat{M}\hat{T} &= \begin{pmatrix} m_{00} & m_{01} & \dots & m_{0n} \\ m_{10} & m_{11} & \dots & m_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n0} & m_{n1} & \dots & m_{nn} \end{pmatrix} \begin{pmatrix} t_{00} & t_{01} & \dots & t_{0n} \\ t_{10} & t_{11} & \dots & t_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n0} & t_{n1} & \dots & t_{nn} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=0}^n m_{0i}t_{i0} & \sum_{i=0}^n m_{0i}t_{i1} & \dots & \sum_{i=0}^n m_{0i}t_{in} \\ \sum_{i=0}^n m_{1i}t_{i0} & \sum_{i=0}^n m_{1i}t_{i1} & \dots & \sum_{i=0}^n m_{1i}t_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^n m_{ni}t_{i0} & \sum_{i=0}^n m_{ni}t_{i1} & \dots & \sum_{i=0}^n m_{ni}t_{in} \end{pmatrix}. \end{aligned} \tag{11}$$

By the same logic, we can find the matrix multiplication of the reverse order,

$$\hat{T}\hat{M} = \begin{pmatrix} \sum_{i=0}^n t_{0i}m_{i0} & \sum_{i=0}^n t_{0i}m_{i1} & \dots & \sum_{i=0}^n t_{0i}m_{in} \\ \sum_{i=0}^n t_{1i}m_{i0} & \sum_{i=0}^n t_{1i}m_{i1} & \dots & \sum_{i=0}^n t_{1i}m_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^n t_{ni}m_{i0} & \sum_{i=0}^n t_{ni}m_{i1} & \dots & \sum_{i=0}^n t_{ni}m_{in} \end{pmatrix}. \tag{12}$$

Crucially, in general, $\hat{M}\hat{T}$ is not equal to $\hat{T}\hat{M}$ because the summation in each entry is not the same. This is in stark contrast to real numbers, for which $ab = ba$ always holds. Let us verify this explicitly.

Exercise 6. Consider two matrices,

$$\hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \textbf{and} \qquad \hat{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate $\hat{A} \cdot \hat{B}$ and $\hat{B} \cdot \hat{A}$ and note that these products are not equal.

We can calculate the amount by which matrices commute using the **commutator**,

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{13}$$

[3] Hats are included on the matrices for later purposes.

Notice that if two matrices commute, $\hat{A}\hat{B} = \hat{B}\hat{A}$, then $[\hat{A}, \hat{B}] = 0$. For two matrices that do not commute, we find that $[\hat{A}, \hat{B}] \neq 0$. Therefore, the commutator is a measure of whether the matrices commute. **From now on, be very careful not to swap matrices around carelessly!**

Again, there is a set of identities to which the commutator adheres:

1. **Anticommutativity**,
$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]; \tag{14}$$
2. **Bilinearity**,
$$[a\hat{A} + b\hat{B}, \hat{C}] = a[\hat{A}, \hat{C}] + b[\hat{B}, \hat{C}]; \tag{15}$$
3. **Leibniz's rule**,
$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}; \tag{16}$$
4. **Jacobi identity**,
$$[[\hat{A}, \hat{B}], \hat{C}] + [[\hat{C}, \hat{A}], \hat{B}] + [[\hat{B}, \hat{C}], \hat{A}] = 0. \tag{17}$$

Exercise 7 (*Tedious*)**. Verify the above identities using the definition of the matrix commutator. Be careful not to swap any matrices since they may not commute.**

Having encountered and verified the Poisson algebra Eq. (5)-(8), one can note the remarkable similarities of the Poisson and commutator algebras. In fact, this simple comparison led Dirac to consider a correspondence between them,

$$\{A, B\} \longleftrightarrow \frac{i}{\hbar}[\hat{A}, \hat{B}], \tag{18}$$

where $A(x, p, t)$ and $B(x, p, t)$ are functions, and $\hat{A}$ and $\hat{B}$ are matrices. [4]

Exercise 8. Using the Dirac rule Eq. (18) **and the expressions of the Poisson brackets from Exercise 1, write immediately down the** $[\hat{x}, \hat{x}]$, $[\hat{p}, \hat{p}]$, $[\hat{x}, \hat{p}]$, $[\hat{x}, \hat{H}]$ **and** $[\hat{p}, \hat{H}]$**.**

With the Dirac correspondence, we may make the bold step to identify classical functions with matrices. In particular, a function $f(x, p, t)$ would be associated with some matrix $\hat{f}$. However, where $f(x, p, t)$ usually depends on the coordinates x and p, and the time t, the matrix operator $\hat{f}$ does not have this information. To ensure that we don't lose any informations, we encode the information of the system in a vector (also called a ket or a state) $|\psi\rangle$ upon which the matrix $\hat{f}$ can act, i.e., $\hat{f}\,|\psi\rangle$ which can be read as a matrix multiplying a vector.

There are generally two (not independent) ways of obtaining measurable information from the operators and states. Firstly, one can look at special vectors $|\psi\rangle$, so-called **eigenstates**, that obey $\hat{f}\,|\psi\rangle = f\,|\psi\rangle$ for some real number f, called the **eigenvalue**. The second way of obtaining measurable values is by considering the expectation value

$$\langle \hat{f} \rangle_\psi \equiv \langle \psi | \hat{f} | \psi \rangle \tag{19}$$

This is a real number (as an inner product) that usually represents a classical approximation of the quantum system. In other words, this would be a value you would measure in the classical counterpart of the system.

[4] Whoops, operators.

Exercise 9. The classical Hamiltonian of the harmonic oscillator is given by,

$$H(x,p,t) = \frac{p^2(t)}{2m} + \frac{m\omega^2 x^2(t)}{2}. \tag{20}$$

Write down immediately the quantum Hamiltonian of the harmonic oscillator.

As demonstrated, translating from a classical Hamiltonian to a quantum Hamiltonian is done through simple hatification. But writing a classical Hamiltonian is one thing, solving it is another. How would one go on solving our quantum Hamiltonian?

Exercise 10. The classical equations of motions of the Hamiltonian are given by the Hamilton equations,

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \textbf{and} \qquad \dot{p} = -\frac{\partial H}{\partial x}. \tag{21}$$

In exercise 1, you wrote these Hamilton equations in terms of Poisson brackets. Postulate the quantum variants of the Hamilton equations using Eq. (18)**.**

We will come back to this.

3. Quantisation

Having replaced all of the classical functions with matrices and operators, how does this lead to quantisation? We have yet to show exactly how this works. In this chapter, you will derive the quantisation of the harmonic oscillator yourself.

In the Exercise 9, you showed that the quantum Hamiltonian of the harmonic oscillator is given by,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}. \tag{22}$$

Solving this equation includes considering the eigenvalue equation $\hat{H}|\psi\rangle = E|\psi\rangle$. This turns out to be a difficult equation, and for this reason, we complete the square and introduce two operators,

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \qquad \text{and} \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right). \tag{23}$$

Exercise 11. Familiarise yourself with $\hat{a}$ and $\hat{a}^\dagger$ by showing that $[\hat{a},\hat{a}^\dagger] = 1$.

This means that we cannot swap the $\hat{a}$ and $\hat{a}^\dagger$ around carelessly.

Exercise 12. It turns out that we can write the Hamiltonian as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{24}$$

Show this by calculating $\hat{a}^\dagger\hat{a}$ explicitly using the results of Exercise 8.

Solving $\hat{H}\left|\psi\right\rangle = E\left|\psi\right\rangle$ will only require us to consider the first term in the brackets since the second term will not affect the state. We define the combination $\hat{N} = \hat{a}^\dagger\hat{a}$ as the **number operator**. We will assume there exist eigenstates of this number operator $\hat{N}\left|n\right\rangle = n\left|n\right\rangle$. Let us investigate some properties.

Exercise 13. Calculate the expressions $\hat{N}(\hat{a}\left|n\right\rangle)$ and $\hat{N}(\hat{a}^\dagger\left|n\right\rangle)$ explicitly.

If you did it correctly, you should have gotten

$$\hat{N}(\hat{a}\left|n\right\rangle) = (n-1)\hat{a}\left|n\right\rangle \qquad \text{and} \qquad \hat{N}(\hat{a}^\dagger\left|n\right\rangle) = (n+1)\hat{a}^\dagger\left|n\right\rangle. \tag{25}$$

Exercise 14. Argue that $\hat{a}\left|n\right\rangle = C\left|n-1\right\rangle$ and $\hat{a}^\dagger\left|n\right\rangle = C'\left|n+1\right\rangle$ for some constants C and C'.

This is a really important step. This ensures that our operators $\hat{a}$ and $\hat{a}^\dagger$ jump between the eigenstates of the number operator $\hat{N}$. We can explicitly find the states by imposing normalisation.

Exercise 15. Determine C and C' by requiring $\left|n\right\rangle$ to be normalised, $\left\langle n|n\right\rangle = 1$.[5]

This completely fixes the states and allows us to extract even more properties of the states.

Exercise 16. What is the lowest value n can go? Keep in mind that $\left|n\right\rangle$ needs to be normalised.

Exercise 17. What are the energy values for the Hamiltonian? Explain the term quantisation.

4. Quantum Evolution

Having worked out an example of quantum mechanics, namely the quantum harmonic oscillator, we found explicit quantisation of energy. We are, however, missing one key component, namely time evolution. In these exercises, you will derive the Schrödinger equation yourself.

Consider two states $\left|\psi(t_0)\right\rangle$ and $\left|\psi(t_1)\right\rangle$ at different times t_0 and t_1. We postulate that time evolution is described by an operator $\hat{\mathcal{U}}(t_1, t_0)$ such that,

$$\left|\psi(t_1)\right\rangle = \hat{\mathcal{U}}(t_1, t_0)\left|\psi(t_0)\right\rangle. \tag{26}$$

By inferring some expected properties, we can write some conditions for our time evolution operator. We have,

1. **Identity:** $\hat{\mathcal{U}}(t_0, t_0) = 1$;
2. **Composition:** $\hat{\mathcal{U}}(t_2, t_0) = \hat{\mathcal{U}}(t_2, t_1)\hat{\mathcal{U}}(t_1, t_0)$;
3. **Inverse:** $\hat{\mathcal{U}}(t_0, t_1)\hat{\mathcal{U}}(t_1, t_0) = \hat{\mathcal{U}}^{-1}(t_1, t_0)\hat{\mathcal{U}}(t_1, t_0) = 1$.

Exercise 18. Explain these properties of the time evolution operator with drawings.

[5] Hint: You might need that the dual of a state $a\left|\psi\right\rangle$ is given by $\left\langle\psi\right|\bar{a}$.

Exercise 19. Using the composition rule, write the evolution of of states from t_0 to t and then to $t + \mathrm{d}t$.

We will focus on the infinitesimal time evolution operator $\hat{\mathcal{U}}(t+\mathrm{d}t, t)$. We can Taylor expand $\hat{\mathcal{U}}(t+\mathrm{d}t, t)$ as,

$$\hat{\mathcal{U}}(t+\mathrm{d}t, t) = 1 - i\hat{\Omega}\mathrm{d}t + \frac{1}{2}\left(-i\hat{\Omega}\mathrm{d}t\right)^2 + \mathcal{O}(\mathrm{d}t)^3, \tag{27}$$

where the prefactor $-i$ is taken for convenience and $\hat{\Omega}$ is an undetermined operator. By taking $\mathrm{d}t$ to be infinitesimal, we may comfortably ignore second order and higher order terms.

Exercise 20. Figure out the units of $\hat{\Omega}$ and compare this to the equation $E = \hbar\omega$ where ω is a frequency. Propose a familiar operator form for $\hat{\Omega}$.

Exercise 21. Plug this expression back into your original one, manipulate the expression and find the Schrödinger equation.

You should find,

$$\frac{\partial}{\partial t}\hat{\mathcal{U}}(t, t_0) = -\frac{i}{\hbar}\hat{H}\hat{\mathcal{U}}(t, t_0). \tag{28}$$

We implore one more tiny detail. We define the backward time-evolution (time-devolution?) as,

$$\hat{\mathcal{U}}^\dagger(t_1, t_0) = \hat{\mathcal{U}}^{-1}(t_1, t_0) = \hat{\mathcal{U}}(t_0, t_1). \tag{29}$$

Mathematically, this † represents taking the transpose, which reflects your matrix over the diagonal, and then taking the complex conjugate of every component, which sends every i to $-i$. Two important details you may assume to be true are $(\hat{A}\hat{B})^T = \hat{B}^T\hat{A}^T$ and $\hat{H}^\dagger = \hat{H}$.[6]

Exercise 22. Take the † (aka transpose and complex conjugate) of either side of the Schrödinger equation to obtain the backwards Schrödinger equation.

You should obtain,

$$\frac{\partial}{\partial t}\hat{\mathcal{U}}^\dagger(t, t_0) = \frac{i}{\hbar}\hat{\mathcal{U}}^\dagger(t, t_0)\hat{H}. \tag{30}$$

5. Different pictures

As mentioned before in Eq. (19), the only measurable variables in quantum mechanics are eigenvalues, or more specifically, expectation values. Recall that for a time-dependent operator $\hat{A}(t)$ and a time-evolved state $|\psi(t)\rangle$, the expectation value $\langle\hat{A}(t)\rangle$ of the operator $\hat{A}(t)$ in the state $|\psi(t)\rangle$ is given by,

$$\langle\hat{A}(t)\rangle_\psi = \langle\psi(t)|\,\hat{A}(t)\,|\psi(t)\rangle\,. \tag{31}$$

[6]The latter reflects that energies are real rather than complex.

Using the definition of a time evolved state,

$$|\psi(t)\rangle = \hat{\mathcal{U}}(t,0)\,|\psi(0)\rangle \equiv \hat{\mathcal{U}}(t)\,|\psi(0)\rangle\,, \tag{32}$$

we can write the expectation value $\langle \hat{A}(t)\rangle$ as,

$$\begin{aligned}\langle \hat{A}(t)\rangle &= \left[\langle\psi(0)|\hat{\mathcal{U}}^\dagger(t)\right]\hat{A}(t)\left[\hat{\mathcal{U}}(t)\,|\psi(0)\rangle\right]\\ &= \langle\psi(0)|\left[\hat{\mathcal{U}}^\dagger(t)A(t)\hat{\mathcal{U}}(t)\right]|\psi(0)\rangle\,.\end{aligned} \tag{33}$$

In the second line, we suggestively rewrote the brackets using the associativity of matrices. This already hints at a new perspective on time dynamics: Instead of evolving the states, we could instead evolve the operators. This is what Heisenberg did, so in his echoing name, we define the Heisenberg operator and states as,

$$\hat{A}_H(t) = \hat{\mathcal{U}}^\dagger(t)\hat{A}(t)\hat{\mathcal{U}}(t) \qquad \text{compared to} \qquad \hat{A}_S(t) = \hat{A}(t), \tag{34}$$

$$|\psi(t)\rangle_H = |\psi(0)\rangle \qquad \text{compared to} \qquad |\psi(t)\rangle_S = \hat{\mathcal{U}}(t)\,|\psi(0)\rangle\,. \tag{35}$$

On the right, we have added the Schrödinger equations for comparison. The subscripts S and H refer to operators and states in the Schrödinger picture and the Heisenberg picture, respectively.

Having defined an operator $\hat{A}_H(t)$ that always depends on time (even when $\hat{A}$ does not), it is only natural to take a time derivative.

Exercise 23 (*Hard*)**. Calculate** $\mathrm{d}\hat{A}_H(t)/\mathrm{d}t$ **explicitly using the product rule and simplify as much as possible. In this, you will need to use the forward and backward Schrödinger equations Eq.** (28) **and Eq.** (30)**.**

You should end up with the Heisenberg equation of motion,

$$\frac{\mathrm{d}\hat{A}_H(t)}{\mathrm{d}t} = -\frac{i}{\hbar}(\hat{A}_H\hat{H}_H - \hat{H}_H\hat{A}_H) + \left(\frac{\partial\hat{A}_S(t)}{\partial t}\right)_H. \tag{36}$$

We can make one more substitution to simplify the expression.

Exercise 24. Use the definition of the commutator to write, [7]

$$\frac{\mathrm{d}\hat{A}_H(t)}{\mathrm{d}t} = -\frac{i}{\hbar}[\hat{A}_H,\hat{H}_H] + \left(\frac{\partial\hat{A}_S(t)}{\partial t}\right)_H. \tag{37}$$

This equation is known as the **Heisenberg equation of motion**.

The Schrödinger picture and the Heisenberg picture are not the only perspectives on the states and operators. There is a third possibility, namely a combination of both. This is called the interaction or Dirac picture. In the last part of this exercise sheet, you will derive this interaction picture yourself.

We begin by splitting a time-dependent Hamiltonian $\hat{H}(t)$ into a time-dependent and a time-independent part as,

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t). \tag{38}$$

We will consider the Schrödinger picture on $\hat{H}_0$ and the Heisenberg picture on $\hat{V}(t)$.

[7] Go back to the first footnote and understand.

Exercise 25. The constant Hamiltonian H_0 should obey the forward and backward Schrödinger equation. Write them down using subscript 0 on the time evolution operators, $\hat{\mathcal{U}}_0(t)$.

We now define the state and the operator in the interaction picture,

$$|\psi(t)\rangle_I = \hat{\mathcal{U}}_0^\dagger(t)\,|\psi(t)\rangle_S \qquad \text{and} \qquad \hat{A}_I = \hat{\mathcal{U}}_0^\dagger(t)\hat{A}_S(t)\hat{\mathcal{U}}_0(t) \tag{39}$$

Notice that we define the interaction picture using the time evolution operator $\hat{\mathcal{U}}_0(t)$ associated with $\hat{H}_0$ and not the time evolution operator $\hat{\mathcal{U}}(t)$ associated with the total Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}(t)$. And alternative way of writing the interaction state would be,

$$|\psi(t)\rangle_S = \hat{\mathcal{U}}_0(t)\,|\psi(t)\rangle_I . \tag{40}$$

We will now see how the interaction state changes with respect to time.

Exercise 26 (*Hard*). **Begin with the Schrödinger equation of the total Hamiltonian,**

$$\frac{\partial}{\partial t}\,|\psi(t)\rangle_S = -\frac{i}{\hbar}\hat{H}\,|\psi(t)\rangle_S , \tag{41}$$

and substitute the definition of $|\psi(t)\rangle_I$, use the product rule and use that $\hat{H} = \hat{H}_0 + \hat{V}(t)$. You should end up with,

$$\frac{\partial}{\partial t}\,|\psi(t)\rangle_I = -\frac{i}{\hbar}\hat{V}_I(t)\,|\psi(t)\rangle_I . \tag{42}$$

This is a Schrödinger-like equation for the interaction states, with the potential acting as a Hamiltonian. Let's see how operators evolve now.

Exercise 27 (*Hard*). **Calculate $\mathrm{d}\hat{A}_I/\mathrm{d}t$ to obtain**

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_I(t) = \frac{i}{\hbar}[\hat{H}_0, \hat{A}_I] + \left(\frac{\partial \hat{A}_S(t)}{\partial t}\right)_I . \tag{43}$$

Thus, the defining equations in the interaction picture are given by,

$$\frac{\partial}{\partial t}\,|\psi(t)\rangle_I = -\frac{i}{\hbar}\hat{V}_I(t)\,|\psi(t)\rangle_I \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_I(t) = \frac{i}{\hbar}[\hat{H}_0, \hat{A}_I] + \left(\frac{\partial \hat{A}_S(t)}{\partial t}\right)_I . \tag{44}$$

To see it is truly a combination of both the Schrödinger picture and the Heisenberg picture, we should have special conditions for either picture to occur.

Exercise 28. What values of $\hat{H}_0$ and $\hat{V}(t)$ should one choose to obtain the Schrödinger picture? And the Heisenberg picture?

Finally, we need to verify that our physics has not changed reality.

Exercise 29. Verify that the expectation value

$$\langle \hat{A}(t)\rangle_\psi = \langle \psi(t)|_I\,\hat{A}_I\,|\psi(t)\rangle_I , \tag{45}$$

reduces to the Schrödinger picture expectation value Eq. (31)**.**

Exercise 30 (*Bonus*). **One operator lost its hat, can you find the operator and the hat?**