Foundation of Scientific Communication:
Slice of Pie

Exercise: Moving a Carousel

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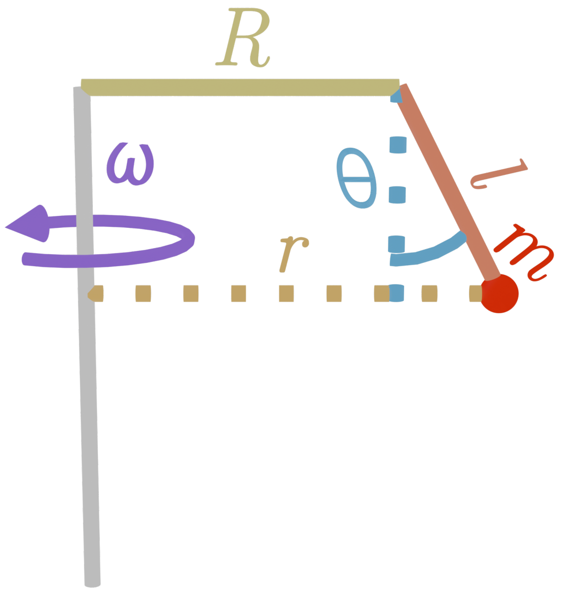
The following paper contains a classical example of a carousel-like system. We will calculate the equations of motion using both Newtonian mechanics and Lagrangian theory.

Let us begin with the Newtonian formalism. As one does, we naturally ignore air friction. Here we have a schematic view of the situation at hand.



Exercise 1. Above is a schematic drawing of the carriage on the carousel, that includes an vertical main axis of rotation, a horizontal bar of length R and a diagonal nonelastic rope of length l at angle θ that connect to a carriage of mass m. The carriage is a length of r away from the main axis. Include all the applicable forces on the carriage in the right directions: gravity $F_g = mg$ downwards, the centrifugal force $F_c = m\omega^2 r$ outwards where ω is the angular velocity of the rotation around the main axis, and not to forget the tension from the rope T in the direction of the rope.

We can now conduct some calculations using the fact that there are no external forces, which means that according to Newton the forces should equate.

Exercise 2. Find a goniometric expression of r in terms of R, l and θ. Write the force balance in the vertical direction to express T in terms of g and θ. Do the same for the horizontal direction and substitute your expressions for r and T into this.

If you have done the exercise properly, you will find the defining equation,

$$g \tan\theta = \omega^2(R + l\sin\theta). \tag{1}$$

This equation tells you that the angle θ at which the carriage rotates is related to the angular velocity ω around the main axis. Solving this equation analytically is unfortunately difficult and outside of the scope of this exercise but you are of course invited to try (for bonus points :D).

This concludes the brief Newtonian derivation to which we will compare our results later. Let us now look at the Lagrangian formalism. To commence, we must recall that the Lagrangian, informally, is the kinetic term minus the potential term of our generalised coordinates. In this case, we take our generalised coordinates to be ϕ and θ, which have generalised coordinates $\dot{\phi}$ and $\dot{\theta}$, respectively.

Let us begin by constructing the kinetic energy. The usual expression $E_k = \frac{1}{2}m\dot{x}^2$ needs to be slightly altered because we are in a rotational system. For a rotation around a main point, the kinetic energy is given by

$$E_k = \frac{1}{2}md^2\dot{\alpha}^2$$

where m is the mass of the rotating object, d is the closest distance to the axis (or point) of rotation, and $\dot{\alpha}$ is the angular velocity in the rotation.

Exercise 3. Draw the possible rotations for the ϕ and θ coordinates. Write down the corresponding equations for the kinetic energies for ϕ and θ.

For potential energy, we need to be a bit more careful. The only potential we encounter is the gravitational potential $E_p = mg\Delta h$.

Exercise 4. Write an expression for the potential energy E_k in terms of θ. You will need to define an reference height. Does the reference height matter?

Having found the expressions for both the kinetic energy E_k and the potential energy E_p, we can now write down the Lagrangian $\mathcal{L} = E_k - E_p$.

Exercise 5. Write down an expression for the Lagrangian $\mathcal{L}$.

If you did everything correctly, you should end up with the Lagrangian,

$$\mathcal{L} = \frac{ml^2\dot{\theta}^2}{2} + \frac{m(R + l\sin\theta)^2\dot{\varphi}^2}{2} - mg(1 - \cos\theta). \tag{2}$$

We quite easily constructed a Lagrangian for the system. To find the equations of motion, we must solve the Euler-Lagrange equations,

$$\frac{\partial\mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}} = 0,$$

where q takes the parameter ϕ or θ.

Exercise 6. Solve the Euler-Lagrange equations for ϕ and interpret your result. What law of conservation could you relate to it? Make a definition such that $\dot{\phi} = \omega$.

By the previous exercise, you found that the carousel rotates in a constant speed since there is no potential energy fuelling the rotation (neither an engine boosting it nor air resistance slowing it down). This allows us to replace every $\dot{\phi}$ with ω, which you may do in the next exercise.

Exercise 7. Solve the Euler-Lagrange equations for θ, substituting all $\dot{\phi} = \omega$. Relate this equation back to the one you found in the Newtonian case. Which one is more general? Interpret your results

Notice that the result from exercise 0.6 is universal.

Exercise 8. If your Lagrangian $\mathcal{L}(q, \dot{q}, t)$ does not explicitly depend on q, what can you say about the Euler-Lagrange equations? Apply this for the case of a free particle $\mathcal{L} = \frac{1}{2}m\dot{q}^2$, because we love free particles.